

# ENGINEERING MATHEMATICS

A Foundation for Electronic, Electrical,  
Communications and Systems Engineers

**FIFTH EDITION**

Anthony Croft • Robert Davison  
Martin Hargreaves • James Flint



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# Engineering Mathematics



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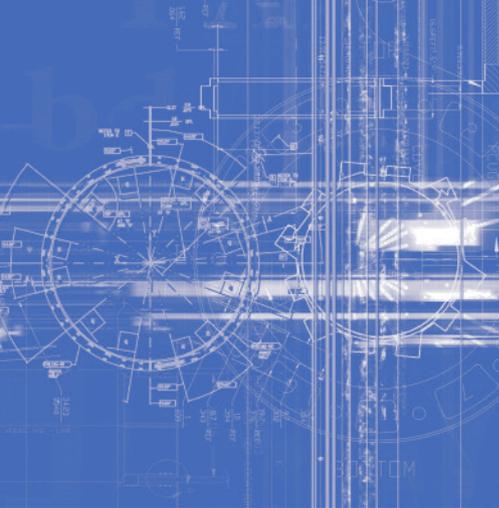
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# Engineering Mathematics

A Foundation for Electronic, Electrical,  
Communications and Systems Engineers

**Anthony Croft**

Loughborough University

**Robert Davison**

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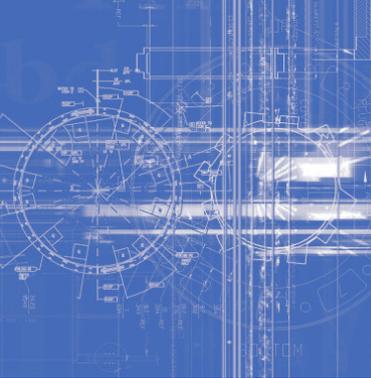
*To Kate, Tom and Harvey – A.C.*

*To Kathy – R.D.*

*To my father and mother – M.H.*

*To Suzanne, Alexandra and Dominic – J.F.*

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# Contents

	Preface	xvii
	Acknowledgements	xix
<b>Chapter 1</b>	<b>Review of algebraic techniques</b>	<b>1</b>
	1.1 Introduction	1
	1.2 Laws of indices	2
	1.3 Number bases	11
	1.4 Polynomial equations	20
	1.5 Algebraic fractions	26
	1.6 Solution of inequalities	33
	1.7 Partial fractions	39
	1.8 Summation notation	46
	Review exercises 1	50
<b>Chapter 2</b>	<b>Engineering functions</b>	<b>54</b>
	2.1 Introduction	54
	2.2 Numbers and intervals	55
	2.3 Basic concepts of functions	56
	2.4 Review of some common engineering functions and techniques	70
	Review exercises 2	113
<b>Chapter 3</b>	<b>The trigonometric functions</b>	<b>115</b>
	3.1 Introduction	115
	3.2 Degrees and radians	116
	3.3 The trigonometric ratios	116
	3.4 The sine, cosine and tangent functions	120
	3.5 The sinc $x$ function	123
	3.6 Trigonometric identities	125
	3.7 Modelling waves using $\sin t$ and $\cos t$	131
	3.8 Trigonometric equations	144
	Review exercises 3	150

<b>Chapter 4</b>	<b>Coordinate systems</b>	<b>154</b>
	4.1 Introduction	154
	4.2 Cartesian coordinate system – two dimensions	154
	4.3 Cartesian coordinate system – three dimensions	157
	4.4 Polar coordinates	159
	4.5 Some simple polar curves	163
	4.6 Cylindrical polar coordinates	166
	4.7 Spherical polar coordinates	170
	Review exercises 4	173
<b>Chapter 5</b>	<b>Discrete mathematics</b>	<b>175</b>
	5.1 Introduction	175
	5.2 Set theory	175
	5.3 Logic	183
	5.4 Boolean algebra	185
	Review exercises 5	197
<b>Chapter 6</b>	<b>Sequences and series</b>	<b>200</b>
	6.1 Introduction	200
	6.2 Sequences	201
	6.3 Series	209
	6.4 The binomial theorem	214
	6.5 Power series	218
	6.6 Sequences arising from the iterative solution of non-linear equations	219
	Review exercises 6	222
<b>Chapter 7</b>	<b>Vectors</b>	<b>224</b>
	7.1 Introduction	224
	7.2 Vectors and scalars: basic concepts	224
	7.3 Cartesian components	232
	7.4 Scalar fields and vector fields	240
	7.5 The scalar product	241
	7.6 The vector product	246
	7.7 Vectors of $n$ dimensions	253
	Review exercises 7	255
<b>Chapter 8</b>	<b>Matrix algebra</b>	<b>257</b>
	8.1 Introduction	257
	8.2 Basic definitions	258

<b>8.3</b>	Addition, subtraction and multiplication	259
<b>8.4</b>	Using matrices in the translation and rotation of vectors	267
<b>8.5</b>	Some special matrices	271
<b>8.6</b>	The inverse of a $2 \times 2$ matrix	274
<b>8.7</b>	Determinants	278
<b>8.8</b>	The inverse of a $3 \times 3$ matrix	281
<b>8.9</b>	Application to the solution of simultaneous equations	283
<b>8.10</b>	Gaussian elimination	286
<b>8.11</b>	Eigenvalues and eigenvectors	294
<b>8.12</b>	Analysis of electrical networks	307
<b>8.13</b>	Iterative techniques for the solution of simultaneous equations	312
<b>8.14</b>	Computer solutions of matrix problems	319
	Review exercises 8	321

## Chapter 9

### Complex numbers **324**

<b>9.1</b>	Introduction	324
<b>9.2</b>	Complex numbers	325
<b>9.3</b>	Operations with complex numbers	328
<b>9.4</b>	Graphical representation of complex numbers	332
<b>9.5</b>	Polar form of a complex number	333
<b>9.6</b>	Vectors and complex numbers	336
<b>9.7</b>	The exponential form of a complex number	337
<b>9.8</b>	Phasors	340
<b>9.9</b>	De Moivre's theorem	344
<b>9.10</b>	Loci and regions of the complex plane	351
	Review exercises 9	354

## Chapter 10

### Differentiation **356**

<b>10.1</b>	Introduction	356
<b>10.2</b>	Graphical approach to differentiation	357
<b>10.3</b>	Limits and continuity	358
<b>10.4</b>	Rate of change at a specific point	362
<b>10.5</b>	Rate of change at a general point	364
<b>10.6</b>	Existence of derivatives	370
<b>10.7</b>	Common derivatives	372
<b>10.8</b>	Differentiation as a linear operator	375
	Review exercises 10	385

## Chapter 11

### Techniques of differentiation **386**

<b>11.1</b>	Introduction	386
-------------	--------------	-----

<b>11.2</b>	Rules of differentiation	386
<b>11.3</b>	Parametric, implicit and logarithmic differentiation	393
<b>11.4</b>	Higher derivatives	400
	Review exercises 11	404

**Chapter 12 Applications of differentiation 406**

<b>12.1</b>	Introduction	406
<b>12.2</b>	Maximum points and minimum points	406
<b>12.3</b>	Points of inflexion	415
<b>12.4</b>	The Newton–Raphson method for solving equations	418
<b>12.5</b>	Differentiation of vectors	423
	Review exercises 12	427

**Chapter 13 Integration 428**

<b>13.1</b>	Introduction	428
<b>13.2</b>	Elementary integration	429
<b>13.3</b>	Definite and indefinite integrals	442
	Review exercises 13	453

**Chapter 14 Techniques of integration 457**

<b>14.1</b>	Introduction	457
<b>14.2</b>	Integration by parts	457
<b>14.3</b>	Integration by substitution	463
<b>14.4</b>	Integration using partial fractions	466
	Review exercises 14	468

**Chapter 15 Applications of integration 471**

<b>15.1</b>	Introduction	471
<b>15.2</b>	Average value of a function	471
<b>15.3</b>	Root mean square value of a function	475
	Review exercises 15	479

**Chapter 16 Further topics in integration 480**

<b>16.1</b>	Introduction	480
<b>16.2</b>	Orthogonal functions	480
<b>16.3</b>	Improper integrals	483
<b>16.4</b>	Integral properties of the delta function	489
<b>16.5</b>	Integration of piecewise continuous functions	491
<b>16.6</b>	Integration of vectors	493
	Review exercises 16	494

<b>Chapter 17</b>	<b>Numerical integration</b>	<b>496</b>
	17.1 Introduction	496
	17.2 Trapezium rule	496
	17.3 Simpson's rule	500
	Review exercises 17	505
<b>Chapter 18</b>	<b>Taylor polynomials, Taylor series and Maclaurin series</b>	<b>507</b>
	18.1 Introduction	507
	18.2 Linearization using first-order Taylor polynomials	508
	18.3 Second-order Taylor polynomials	513
	18.4 Taylor polynomials of the $n$ th order	517
	18.5 Taylor's formula and the remainder term	521
	18.6 Taylor and Maclaurin series	524
	Review exercises 18	532
<b>Chapter 19</b>	<b>Ordinary differential equations I</b>	<b>534</b>
	19.1 Introduction	534
	19.2 Basic definitions	535
	19.3 First-order equations: simple equations and separation of variables	540
	19.4 First-order linear equations: use of an integrating factor	547
	19.5 Second-order linear equations	558
	19.6 Constant coefficient equations	560
	19.7 Series solution of differential equations	584
	19.8 Bessel's equation and Bessel functions	587
	Review exercises 19	601
<b>Chapter 20</b>	<b>Ordinary differential equations II</b>	<b>603</b>
	20.1 Introduction	603
	20.2 Analogue simulation	603
	20.3 Higher order equations	606
	20.4 State-space models	609
	20.5 Numerical methods	615
	20.6 Euler's method	616
	20.7 Improved Euler method	620
	20.8 Runge–Kutta method of order 4	623
	Review exercises 20	626
<b>Chapter 21</b>	<b>The Laplace transform</b>	<b>627</b>
	21.1 Introduction	627
	21.2 Definition of the Laplace transform	628

<b>21.3</b>	Laplace transforms of some common functions	629
<b>21.4</b>	Properties of the Laplace transform	631
<b>21.5</b>	Laplace transform of derivatives and integrals	635
<b>21.6</b>	Inverse Laplace transforms	638
<b>21.7</b>	Using partial fractions to find the inverse Laplace transform	641
<b>21.8</b>	Finding the inverse Laplace transform using complex numbers	643
<b>21.9</b>	The convolution theorem	647
<b>21.10</b>	Solving linear constant coefficient differential equations using the Laplace transform	649
<b>21.11</b>	Transfer functions	659
<b>21.12</b>	Poles, zeros and the $s$ plane	668
<b>21.13</b>	Laplace transforms of some special functions	675
	Review exercises 21	678
<b>Chapter 22</b>		
	<b>Difference equations and the <math>z</math> transform</b>	<b>681</b>
<b>22.1</b>	Introduction	681
<b>22.2</b>	Basic definitions	682
<b>22.3</b>	Rewriting difference equations	686
<b>22.4</b>	Block diagram representation of difference equations	688
<b>22.5</b>	Design of a discrete-time controller	693
<b>22.6</b>	Numerical solution of difference equations	695
<b>22.7</b>	Definition of the $z$ transform	698
<b>22.8</b>	Sampling a continuous signal	702
<b>22.9</b>	The relationship between the $z$ transform and the Laplace transform	704
<b>22.10</b>	Properties of the $z$ transform	709
<b>22.11</b>	Inversion of $z$ transforms	715
<b>22.12</b>	The $z$ transform and difference equations	718
	Review exercises 22	720
<b>Chapter 23</b>		
	<b>Fourier series</b>	<b>722</b>
<b>23.1</b>	Introduction	722
<b>23.2</b>	Periodic waveforms	723
<b>23.3</b>	Odd and even functions	726
<b>23.4</b>	Orthogonality relations and other useful identities	732
<b>23.5</b>	Fourier series	733
<b>23.6</b>	Half-range series	745
<b>23.7</b>	Parseval's theorem	748
<b>23.8</b>	Complex notation	749
<b>23.9</b>	Frequency response of a linear system	751
	Review exercises 23	755

<b>Chapter 24</b>	<b>The Fourier transform</b>	<b>757</b>
	24.1 Introduction	757
	24.2 The Fourier transform – definitions	758
	24.3 Some properties of the Fourier transform	761
	24.4 Spectra	766
	24.5 The $t$ – $\omega$ duality principle	768
	24.6 Fourier transforms of some special functions	770
	24.7 The relationship between the Fourier transform and the Laplace transform	772
	24.8 Convolution and correlation	774
	24.9 The discrete Fourier transform	783
	24.10 Derivation of the d.f.t.	787
	24.11 Using the d.f.t. to estimate a Fourier transform	790
	24.12 Matrix representation of the d.f.t.	792
	24.13 Some properties of the d.f.t.	793
	24.14 The discrete cosine transform	795
	24.15 Discrete convolution and correlation	801
	Review exercises 24	821
<b>Chapter 25</b>	<b>Functions of several variables</b>	<b>823</b>
	25.1 Introduction	823
	25.2 Functions of more than one variable	823
	25.3 Partial derivatives	825
	25.4 Higher order derivatives	829
	25.5 Partial differential equations	832
	25.6 Taylor polynomials and Taylor series in two variables	835
	25.7 Maximum and minimum points of a function of two variables	841
	Review exercises 25	846
<b>Chapter 26</b>	<b>Vector calculus</b>	<b>849</b>
	26.1 Introduction	849
	26.2 Partial differentiation of vectors	849
	26.3 The gradient of a scalar field	851
	26.4 The divergence of a vector field	856
	26.5 The curl of a vector field	859
	26.6 Combining the operators grad, div and curl	861
	26.7 Vector calculus and electromagnetism	864
	Review exercises 26	865

<b>Chapter 27</b>	<b>Line integrals and multiple integrals</b>	<b>867</b>
	27.1 Introduction	867
	27.2 Line integrals	867
	27.3 Evaluation of line integrals in two dimensions	871
	27.4 Evaluation of line integrals in three dimensions	873
	27.5 Conservative fields and potential functions	875
	27.6 Double and triple integrals	880
	27.7 Some simple volume and surface integrals	889
	27.8 The divergence theorem and Stokes' theorem	895
	27.9 Maxwell's equations in integral form	899
	Review exercises 27	901
<b>Chapter 28</b>	<b>Probability</b>	<b>903</b>
	28.1 Introduction	903
	28.2 Introducing probability	904
	28.3 Mutually exclusive events: the addition law of probability	909
	28.4 Complementary events	913
	28.5 Concepts from communication theory	915
	28.6 Conditional probability: the multiplication law	919
	28.7 Independent events	925
	Review exercises 28	930
<b>Chapter 29</b>	<b>Statistics and probability distributions</b>	<b>933</b>
	29.1 Introduction	933
	29.2 Random variables	934
	29.3 Probability distributions – discrete variable	935
	29.4 Probability density functions – continuous variable	936
	29.5 Mean value	938
	29.6 Standard deviation	941
	29.7 Expected value of a random variable	943
	29.8 Standard deviation of a random variable	946
	29.9 Permutations and combinations	948
	29.10 The binomial distribution	953
	29.11 The Poisson distribution	957
	29.12 The uniform distribution	961
	29.13 The exponential distribution	962
	29.14 The normal distribution	963
	29.15 Reliability engineering	970
	Review exercises 29	977

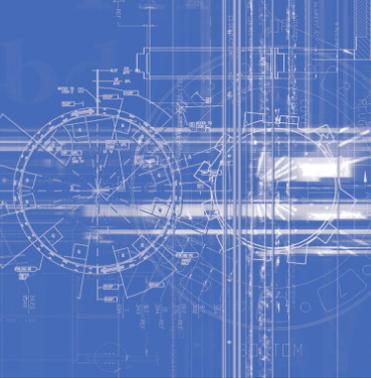
<b>Appendix I</b>	Representing a continuous function and a sequence as a sum of weighted impulses	979
<b>Appendix II</b>	The Greek alphabet	981
<b>Appendix III</b>	SI units and prefixes	982
<b>Appendix IV</b>	The binomial expansion of $\left(\frac{n-N}{n}\right)^n$	982
<b>Index</b>		<b>983</b>

## Lecturer Resources

For password-protected online resources tailored to support the use of this textbook in teaching, please visit [www.pearsoned.co.uk/croft](http://www.pearsoned.co.uk/croft)



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# Preface

## Audience

This book has been written to serve the mathematical needs of students engaged in a first course in engineering at degree level. It is primarily aimed at students of electronic, electrical, communications and systems engineering. Systems engineering typically encompasses areas such as manufacturing, control and production engineering. The textbook will also be useful for engineers who wish to engage in self-study and continuing education.

## Motivation

Engineers are called upon to analyse a variety of engineering systems, which can be anything from a few electronic components connected together through to a complete factory. The analysis of these systems benefits from the intelligent application of mathematics. Indeed, many cannot be analysed without the use of mathematics. Mathematics is the language of engineering. It is essential to understand how mathematics works in order to master the complex relationships present in modern engineering systems and products.

## Aims

There are two main aims of the book. Firstly, we wish to provide an accessible, readable introduction to engineering mathematics at degree level. The second aim is to encourage the integration of engineering and mathematics.

## Content

The first three chapters include a review of some important functions and techniques that the reader may have met in previous courses. This material ensures that the book is self-contained and provides a convenient reference.

Traditional topics in algebra, trigonometry and calculus have been covered. Also included are chapters on set theory, sequences and series, Boolean algebra, logic, difference equations and the  $z$  transform. The importance of signal processing techniques is reflected by a thorough treatment of integral transform methods. Thus the Laplace,  $z$  and Fourier transforms have been given extensive coverage.

In the light of feedback from readers, new topics and new examples have been added in the fifth edition. Recognizing that motivation comes from seeing the applicability of mathematics we have focused mainly on the enhancement of the range of applied examples. These include topics on the discrete cosine transform, image processing, applications in music technology, communications engineering and frequency modulation.

## Style

The style of the book is to develop and illustrate mathematical concepts through examples. We have tried throughout to adopt an informal approach and to describe mathematical processes using everyday language. Mathematical ideas are often developed by examples rather than by using abstract proof, which has been kept to a minimum. This reflects the authors' experience that engineering students learn better from practical examples, rather than from formal abstract development. We have included many engineering examples and have tried to make them as free-standing as possible to keep the necessary engineering prerequisites to a minimum. The engineering examples, which have been carefully selected to be relevant, informative and modern, range from short illustrative examples through to complete sections which can be regarded as case studies. A further benefit is the development of the link between mathematics and the physical world. An appreciation of this link is essential if engineers are to take full advantage of engineering mathematics. The engineering examples make the book more colourful and, more importantly, they help develop the ability to see an engineering problem and translate it into a mathematical form so that a solution can be obtained. This is one of the most difficult skills that an engineer needs to acquire. The ability to manipulate mathematical equations is by itself insufficient. It is sometimes necessary to derive the equations corresponding to an engineering problem. Interpretation of mathematical solutions in terms of the physical variables is also essential. Engineers cannot afford to get lost in mathematical symbolism.

## Format

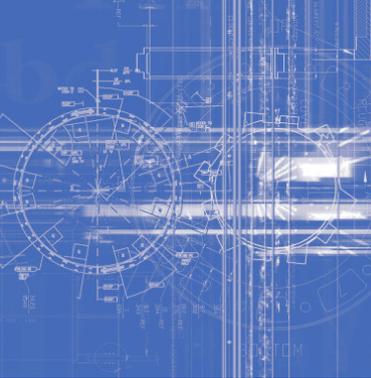
Important results are highlighted for easy reference. Exercises and solutions are provided at the end of most sections; it is essential to attempt these as the only way to develop competence and understanding is through practice. A further set of review exercises is provided at the end of each chapter. In addition some sections include exercises that are intended to be carried out on a computer using a technical computing language such as MATLAB<sup>®</sup>, GNU Octave, Mathematica or Python<sup>®</sup>. The MATLAB<sup>®</sup> command syntax is supported in several software packages as well as MATLAB<sup>®</sup> itself and will be used throughout the book.

## Supplements

A comprehensive Solutions Manual is obtainable free of charge to lecturers using this textbook. It is also available for download via the web at [www.pearsoned.co.uk/croft](http://www.pearsoned.co.uk/croft).

Finally we hope you will come to share our enthusiasm for engineering mathematics and enjoy the book.

*Anthony Croft  
Robert Davison  
Martin Hargreaves  
James Flint  
March 2017*



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## Tables

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# 1 Review of algebraic techniques

## Contents

1.1	Introduction	1
1.2	Laws of indices	2
1.3	Number bases	11
1.4	Polynomial equations	20
1.5	Algebraic fractions	26
1.6	Solution of inequalities	33
1.7	Partial fractions	39
1.8	Summation notation	46
	Review exercises 1	50

## 1.1 INTRODUCTION

This chapter introduces some algebraic techniques which are commonly used in engineering mathematics. For some readers this may be revision. Section 1.2 examines the laws of indices. These laws are used throughout engineering mathematics. Section 1.3 looks at number bases. Section 1.4 looks at methods of solving polynomial equations. Section 1.5 examines algebraic fractions, while Section 1.6 examines the solution of inequalities. Section 1.7 looks at partial fractions. The chapter closes with a study of summation notation.

Computers are used extensively in all engineering disciplines to perform calculations. Some of the examples provided in this book make use of the technical computing language **MATLAB**<sup>®</sup>, which is commonly used in both an academic and industrial setting.

Because **MATLAB**<sup>®</sup> and many other similar languages are designed to compute not just with single numbers but with entire sequences of numbers at the same time, data is entered in the form of **arrays**. These are multi-dimensional objects. Two particular types of array are **vectors** and **matrices** which are studied in detail in Chapters 7 and 8.

Apart from being able to perform basic mathematical operations with vectors and matrices, MATLAB<sup>®</sup> has, in addition, a vast range of built-in computational functions which are straightforward to use but nevertheless are very powerful. Many of these high-level functions are accessible by passing data to them in the form of vectors and matrices. A small number of these special functions are used and explained in this text. However, to get the most out of a technical computing language it is necessary to develop a good understanding of what the software can do and to make regular reference to the manual.

## 1.2 LAWS OF INDICES

Consider the product  $6 \times 6 \times 6 \times 6 \times 6$ . This may be written more compactly as  $6^5$ . We call 5 the **index** or **power**. The **base** is 6. Similarly,  $y \times y \times y \times y$  may be written as  $y^4$ . Here the base is  $y$  and the index is 4.

**Example 1.1** Write the following using index notation:

(a)  $(-2)(-2)(-2)$     (b)  $4.4.4.5.5$     (c)  $\frac{yyy}{xxxx}$     (d)  $\frac{aa(-a)(-a)}{bb(-b)}$

**Solution** (a)  $(-2)(-2)(-2)$  may be written as  $(-2)^3$ .

(b)  $4.4.4.5.5$  may be written as  $4^3 5^2$ .

(c)  $\frac{yyy}{xxxx}$  may be written as  $\frac{y^3}{x^4}$ .

(d) Note that  $(-a)(-a) = aa$  since the product of two negative quantities is positive. So  $aa(-a)(-a) = aaaa = a^4$ . Also  $bb(-b) = -bbb = -b^3$ . Hence

$$\frac{aa(-a)(-a)}{bb(-b)} = \frac{a^4}{-b^3} = -\frac{a^4}{b^3}$$

**Example 1.2** Evaluate

(a)  $7^3$     (b)  $(-3)^3$     (c)  $2^3(-3)^4$

**Solution** (a)  $7^3 = 7.7.7 = 343$

(b)  $(-3)^3 = (-3)(-3)(-3) = -27$

(c)  $2^3(-3)^4 = 8(81) = 648$

Most scientific calculators have an  $x^y$  button to enable easy calculation of expressions of a similar form to those in Example 1.2.

### 1.2.1 Multiplying expressions involving indices

Consider the product  $(6^2)(6^3)$ . We may write this as

$$(6^2)(6^3) = (6.6)(6.6.6) = 6^5$$

So

$$6^2 6^3 = 6^5$$

This illustrates the **first law of indices** which is

$$a^m a^n = a^{m+n}$$

When expressions with the same base are multiplied, the indices are added.

**Example 1.3** Simplify each of the following expressions:

(a)  $3^9 3^{10}$       (b)  $4^3 4^4 4^6$       (c)  $x^3 x^6$       (d)  $y^4 y^2 y^3$

**Solution**

(a)  $3^9 3^{10} = 3^{9+10} = 3^{19}$   
 (b)  $4^3 4^4 4^6 = 4^{3+4+6} = 4^{13}$   
 (c)  $x^3 x^6 = x^{3+6} = x^9$   
 (d)  $y^4 y^2 y^3 = y^{4+2+3} = y^9$

### Engineering application 1.1

#### Power dissipation in a resistor

The **resistor** is one of the three fundamental electronic components. The other two are the capacitor and the inductor, which we will meet later. The role of the resistor is to reduce the current flow within the branch of a circuit for a given voltage. As current flows through the resistor, electrical energy is converted into heat. Because the energy is lost from the circuit and is effectively wasted, it is termed **dissipated** energy. The rate of energy dissipation is known as the power,  $P$ , and is given by

$$P = I^2 R \quad (1.1)$$

where  $I$  is the current flowing through the resistor and  $R$  is the resistance value. Note that the current is raised to the power 2. Note that power,  $P$ , is measured in watts; current,  $I$ , is measured in amps; and resistance,  $R$ , is measured in ohms.

There is an alternative formula for power dissipation in a resistor that uses the voltage,  $V$ , across the resistor. To obtain this alternative formula we need to use **Ohm's law**, which states that the voltage across a resistor,  $V$ , and the current passing through it, are related by the formula

$$V = IR \quad (1.2)$$

From Equation (1.2) we see that

$$I = \frac{V}{R} \quad (1.3)$$

Combining Equations (1.1) and (1.3) gives

$$P = \left(\frac{V}{R}\right)^2 R = \frac{V}{R} \cdot \frac{V}{R} \cdot R = \frac{V^2}{R}$$



Note that in this formula for  $P$ , the voltage is raised to the power 2. Note an important consequence of this formula is that doubling the voltage, while keeping the resistance fixed, results in the power dissipation increasing by a factor of 4, that is  $2^2$ . Also trebling the voltage, for a fixed value of resistance, results in the power dissipation increasing by a factor of 9, that is  $3^2$ .

Similar considerations can be applied to Equation 1.1. For a fixed value of resistance, doubling the current results in the power dissipation increasing by a factor of 4, and trebling the current results in the power dissipation increasing by a factor of 9.

Consider the product  $3(3^3)$ . Now

$$3(3^3) = 3(3.3.3) = 3^4$$

Also, using the first law of indices we see that  $3^1 3^3 = 3^4$ . This suggests that 3 is the same as  $3^1$ . This illustrates the general rule:

$$a = a^1$$

Raising a number to the power 1 leaves the number unchanged.

**Example 1.4** Simplify (a)  $5^6 5$  (b)  $x^3 x x^2$

**Solution** (a)  $5^6 5 = 5^{6+1} = 5^7$  (b)  $x^3 x x^2 = x^{3+1+2} = x^6$

## 1.2.2 Dividing expressions involving indices

Consider the expression  $\frac{4^5}{4^3}$ :

$$\begin{aligned} \frac{4^5}{4^3} &= \frac{4.4.4.4.4}{4.4.4} \\ &= 4.4 \quad \text{by cancelling 4s} \\ &= 4^2 \end{aligned}$$

This serves to illustrate the **second law of indices** which is

$$\frac{a^m}{a^n} = a^{m-n}$$

When expressions with the same base are divided, the indices are subtracted.

**Example 1.5** Simplify

(a)  $\frac{5^9}{5^7}$  (b)  $\frac{(-2)^{16}}{(-2)^{13}}$  (c)  $\frac{x^9}{x^5}$  (d)  $\frac{y^6}{y}$

**Solution** (a)  $\frac{5^9}{5^7} = 5^{9-7} = 5^2$

(b)  $\frac{(-2)^{16}}{(-2)^{13}} = (-2)^{16-13} = (-2)^3$

$$(c) \frac{x^9}{x^5} = x^{9-5} = x^4$$

$$(d) \frac{y^6}{y} = y^{6-1} = y^5$$

Consider the expression  $\frac{2^3}{2^3}$ . Using the second law of indices we may write

$$\frac{2^3}{2^3} = 2^{3-3} = 2^0$$

But, clearly,  $\frac{2^3}{2^3} = 1$ , and so  $2^0 = 1$ . This illustrates the general rule:

$$a^0 = 1$$

Any expression raised to the power 0 is 1.

### 1.2.3 Negative indices

Consider the expression  $\frac{4^3}{4^5}$ . We can write this as

$$\frac{4^3}{4^5} = \frac{4 \cdot 4 \cdot 4}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4} = \frac{1}{4 \cdot 4} = \frac{1}{4^2}$$

Alternatively, using the second law of indices we have

$$\frac{4^3}{4^5} = 4^{3-5} = 4^{-2}$$

So we see that

$$4^{-2} = \frac{1}{4^2}$$

Thus we are able to interpret negative indices. The sign of an index changes when the expression is inverted. In general we can state

$$a^{-m} = \frac{1}{a^m} \quad a^m = \frac{1}{a^{-m}}$$

**Example 1.6** Evaluate the following:

$$(a) 3^{-2} \quad (b) \frac{2}{4^{-3}} \quad (c) 3^{-1} \quad (d) (-3)^{-2} \quad (e) \frac{6^{-3}}{6^{-2}}$$

**Solution**

$$(a) 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$(b) \frac{2}{4^{-3}} = 2(4^3) = 2(64) = 128$$

$$(c) 3^{-1} = \frac{1}{3^1} = \frac{1}{3}$$

$$(d) (-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$$

$$(e) \frac{6^{-3}}{6^{-2}} = 6^{-3-(-2)} = 6^{-1} = \frac{1}{6^1} = \frac{1}{6}$$

**Example 1.7** Write the following expressions using only positive indices:

(a)  $x^{-4}$     (b)  $3x^{-4}$     (c)  $\frac{x^{-2}}{y^{-2}}$     (d)  $3x^{-2}y^{-3}$

**Solution**

(a)  $x^{-4} = \frac{1}{x^4}$

(b)  $3x^{-4} = \frac{3}{x^4}$

(c)  $\frac{x^{-2}}{y^{-2}} = x^{-2}y^2 = \frac{y^2}{x^2}$

(d)  $3x^{-2}y^{-3} = \frac{3}{x^2y^3}$

## Engineering application 1.2

### Power density of a signal transmitted by a radio antenna

A **radio antenna** is a device that is used to convert electrical energy into electromagnetic radiation, which is then transmitted to distant points.

An ideal theoretical point source radio antenna which radiates the same power in all directions is termed an **isotropic** antenna. When it transmits a radio wave, the wave spreads out equally in all directions, providing there are no obstacles to block the expansion of the wave. The power generated by the antenna is uniformly distributed on the surface of an expanding sphere of area,  $A$ , given by

$$A = 4\pi r^2$$

where  $r$  is the distance from the generating antenna to the wave front.

The **power density**,  $S$ , provides an indication of how much of the signal can potentially be received by another antenna placed at a distance  $r$ . The actual power received depends on the effective area or aperture of the antenna, which is usually expressed in units of  $\text{m}^2$ .

Electromagnetic field exposure limits for humans are sometimes specified in terms of a power density. The closer a person is to the transmitter, the higher the power density will be. So a safe distance needs to be determined.

The power density is the ratio of the power transmitted,  $P_t$ , to the area over which it is spread

$$S = \frac{\text{power transmitted}}{\text{area}} = \frac{P_t}{4\pi r^2} = \frac{P_t}{4\pi} r^{-2} \text{ W m}^{-2}$$

Note that  $r$  in this equation has a **negative index**. This type of relationship is known as an **inverse square law** and is found commonly in science and engineering.

Note that if the distance,  $r$ , is doubled, then the area,  $A$ , increases by a factor of 4 (i.e.  $2^2$ ). If the distance is trebled, the area increases by a factor of 9 (i.e.  $3^2$ ) and so on. This means that as the distance from the antenna doubles, the power density,  $S$ , decreases to a quarter of its previous value; if the distance trebles then the power density is only a ninth of its previous value.

### 1.2.4 Multiple indices

Consider the expression  $(4^3)^2$ . This may be written as

$$(4^3)^2 = 4^3 \cdot 4^3 = 4^{3+3} = 4^6$$

This illustrates the **third law of indices** which is

$$(a^m)^n = a^{mn}$$

Note that the indices  $m$  and  $n$  have been multiplied.

**Example 1.8** Write the following expressions using a single index:

(a)  $(3^2)^4$       (b)  $(7^{-2})^3$       (c)  $(x^2)^{-3}$       (d)  $(x^{-2})^{-3}$

**Solution**

(a)  $(3^2)^4 = 3^{2 \times 4} = 3^8$   
 (b)  $(7^{-2})^3 = 7^{-2 \times 3} = 7^{-6}$   
 (c)  $(x^2)^{-3} = x^{2 \times (-3)} = x^{-6}$   
 (d)  $(x^{-2})^{-3} = x^{-2 \times -3} = x^6$

Consider the expression  $(2^4 5^2)^3$ . We see that

$$\begin{aligned} (2^4 5^2)^3 &= (2^4 5^2)(2^4 5^2)(2^4 5^2) \\ &= 2^4 2^4 2^4 5^2 5^2 5^2 \\ &= 2^{12} 5^6 \end{aligned}$$

This illustrates a generalization of the third law of indices which is

$$(a^m b^n)^k = a^{mk} b^{nk}$$

**Example 1.9** Remove the brackets from

(a)  $(2x^2)^3$       (b)  $(-3y^4)^2$       (c)  $(x^{-2}y)^3$

**Solution**

(a)  $(2x^2)^3 = (2^1 x^2)^3 = 2^3 x^6 = 8x^6$   
 (b)  $(-3y^4)^2 = (-3)^2 y^8 = 9y^8$   
 (c)  $(x^{-2}y)^3 = x^{-6}y^3$

### Engineering application 1.3

#### Radar scattering

It has already been shown in Engineering application 1.2 that the power density of an isotropic transmitter of radio waves is

$$S = \frac{P_t}{4\pi} r^{-2} \text{ W m}^{-2}$$



It is possible to use radio waves to detect distant objects. The technique involves transmitting a radio signal, which is then reflected back when it strikes a target. This weak reflected signal is then picked up by a receiving antenna, thus allowing a number of properties of the target to be deduced, such as its angular position and distance from the transmitter. This system is known as **radar**, which was originally an acronym standing for **RA**dio **D**etection **A**nd **R**anging.

When the wave hits the target it produces a quantity of reflected power. The power depends upon the object's **radar cross-section** (RCS), normally denoted by the Greek lower case letter sigma,  $\sigma$ , and having units of  $\text{m}^2$ . The power reflected at the object,  $P_r$ , is given by

$$P_r = S\sigma = \frac{P_t\sigma}{4\pi}r^{-2} \text{ W}$$

Some military aircraft use special techniques to minimize the RCS in order to reduce the amount of power they reflect and hence minimize the chance of being detected.

If the reflected power at the target is assumed to spread spherically, when it returns to the transmitter position it will have the power density,  $S_r$ , given by

$$S_r = \frac{\text{power reflected at target}}{\text{area}} = \frac{P_r}{4\pi}r^{-2} \text{ W m}^{-2}$$

Substituting for the reflected power,  $P_r$ , gives

$$\begin{aligned} S_r &= \frac{\text{power reflected at target}}{\text{area}} = \frac{\left(\frac{P_t\sigma}{4\pi}r^{-2}\right)}{4\pi}r^{-2} = \frac{P_t\sigma}{4\pi \times 4\pi} (r^{-2})^2 \\ &= \frac{P_t\sigma}{(4\pi)^2}r^{-4} \text{ W m}^{-2} \end{aligned}$$

Note that the product of the two  $r^{-2}$  terms has been calculated using the third law of indices.

This example illustrates one of the main challenges with radar design which is that the power density returned by a distant object is very much smaller than the transmitted power, even for targets with a large RCS. For theoretical isotropic antennas, the received power density depends upon the factor  $r^{-4}$ . This factor diminishes rapidly for large values of  $r$ , that is, as the object being detected gets further away.

In practice, the transmit antennas used are not isotropic but **directive** and often scan the area of interest. They also make use of receive antennas with a large effective area which can produce a viable signal from the small reflected power densities.

### 1.2.5 Fractional indices

The third law of indices states that  $(a^m)^n = a^{mn}$ . If we take  $a = 2$ ,  $m = \frac{1}{2}$  and  $n = 2$  we obtain

$$(2^{1/2})^2 = 2^1 = 2$$

So when  $2^{1/2}$  is squared, the result is 2. Thus,  $2^{1/2}$  is a square root of 2. Each positive number has two square roots and so

$$2^{1/2} = \sqrt{2} = \pm 1.4142\dots$$

Similarly

$$(2^{1/3})^3 = 2^1 = 2$$

so that  $2^{1/3}$  is a cube root of 2:

$$2^{1/3} = \sqrt[3]{2} = 1.2599 \dots$$

In general  $2^{1/n}$  is an  $n$ th root of 2. The general law states

$$x^{1/n} \text{ is an } n\text{th root of } x$$

**Example 1.10** Write the following using a single positive index:

(a)  $(3^{-2})^{1/4}$       (b)  $x^{2/3}x^{5/3}$       (c)  $yy^{-2/5}$       (d)  $\sqrt{k^3}$

**Solution**

(a)  $(3^{-2})^{1/4} = 3^{-2 \times \frac{1}{4}} = 3^{-1/2} = \frac{1}{3^{1/2}}$   
 (b)  $x^{2/3}x^{5/3} = x^{2/3+5/3} = x^{7/3}$   
 (c)  $yy^{-2/5} = y^1y^{-2/5} = y^{1-2/5} = y^{3/5}$   
 (d)  $\sqrt{k^3} = (k^3)^{1/2} = k^{3 \times \frac{1}{2}} = k^{3/2}$

**Example 1.11** Evaluate

(a)  $8^{1/3}$       (b)  $8^{2/3}$       (c)  $8^{-1/3}$       (d)  $8^{-2/3}$       (e)  $8^{4/3}$

**Solution** We note that 8 may be written as  $2^3$ .

(a)  $8^{1/3} = (2^3)^{1/3} = 2^1 = 2$   
 (b)  $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$   
 (c)  $8^{-1/3} = \frac{1}{8^{1/3}} = \frac{1}{2}$   
 (d)  $8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{4}$   
 (e)  $8^{4/3} = (8^{1/3})^4 = 2^4 = 16$

## Engineering application 1.4

### Skin depth in a radial conductor

When an alternating current signal travels along a conductor, such as a copper wire, most of the current is found near the surface of the conductor. Nearer to the centre of the conductor, the current diminishes. The depth of penetration of the signal, termed the **skin depth**, into the conductor depends on the frequency of the signal. Skin depth, illustrated in Figure 1.1, is defined as the depth at which the current

